Test 4 Solutions

1. Let (A_n/B_n) be the sequence of continued fraction convergents to the real irrational number α. Of the following statements:

(1) the sequence (A_n/B_n) is monotone increasing;

(2) the sequence of differences $(\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}})$ has alternately positive and negative terms;

(3) $A_{2n} < \alpha B_{2n}$ for all $n \geq 0$;

the ones which are **true** are

(a) (1) and (2) only; (b) (1) and (3) only; (c) (2) and (3) only; (d) (2) only.

Solution: Statement (1) is **false**: although the even convergents form an increasing sequence, the odd convergents form a decreasing sequence (see (2.8)(b,c) in Workbook 4.), so overall the sequence is neither increasing nor decreasing. Statement (2) is **true** by (2.8)(a). Statement (3) is **true**, since α is the limit of the increasing sequence A_{2n}/B_{2n}, so each term of this sequence is $< \alpha$. Hence the answer is **(c)**. (The logicians amongst you will point out that (1) **false** and (3) **true** already imply that **(c)** is the only possible answer!)

2. The polynomial $[q_0\,,\,q_1\,,\,q_2\,,\,\ldots\,,q_t]$, which describes the numerator of the continued fraction $q_0 + \frac{1}{q_1+} \cdots \frac{1}{q_t}$, is a sum of terms called monomials. For example, $[q_0\,,\,q_1]$ is the sum of the two monomials q_0q_1 and 1 (the constant term counts as one of the monomials). The number of monomials appearing in the sum for $[q_0\,,\,q_1\,,\,q_2\,,\,q_3\,,\,q_4]$ is equal to

(a) 8 (b) 10 (c) 13 (d) none of these

Solution: The calculation is done on page 12 of Workbook 4 (Part (b) of Worked Example 1.11). It runs $[q_0, q_1, q_2, q_3, q_4] = q_0q_1q_2q_3q_4 + q_0q_1q_2 + q_0q_1q_4 + q_0q_3q_4 + q_2q_3q_4 + q_0 + q_2 + q_4$. Thus there are 8 monomials in the sum and so **(a)** is the correct answer. (Can you see why in general the answer is the t'th Fibonacci number?)

3. When $\dfrac{2013}{2005}$ is written as a (finite) continued fraction $q_0 + \dfrac{1}{q_1+}\dfrac{1}{q_2+}\cdots\dfrac{1}{q_t}$ (with $q_t > 1$),

(a) $t = 3$ and $q_t = 3$ (b) $t = 5$ and $q_t = 2$ (c) $t = 7$ and $q_t = 2$ (d) none of these.

Solution: Applying the Euclidean Algorithm to 2013 and 2005, the remainder sequence is $2013, 2005, 8, 5, 3, 2, 1$ with quotients $1, 250, 1, 1, 1, 1$. Hence $t = 4$ and $q_t = 2$, answer **(b)**: remember that the q_i are numbered from $i = 0$! The CF expansion is $1 + \frac{1}{250+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2}$.

4. Let $\left(2 + \frac{1}{2+}\frac{1}{2}\right) + \left(4 + \frac{1}{4+}\frac{1}{4}\right) = q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots\frac{1}{q_{t-1}+}\frac{1}{q_t}$ (where $q_t > 1$). Then

(a) $t = 5$ and $q_t = 5$ (b) $t = 6$ and $q_t = 7$ (c) $t = 7$ and $q_t = 2$ (d) $t = 8$ and $q_t = 5$

Solution: First expand $2 + \frac{1}{2+}\frac{1}{2} = \frac{12}{5}$ and $4 + \frac{1}{4+}\frac{1}{4} = \frac{72}{17}$, so we need the CF expansion of $\frac{12}{5} + \frac{72}{17} = \frac{564}{85}$. The remainder sequence is $564, 85, 54, 31, 23, 8, 7, 1$ with quotient sequence $6, 1, 1, 1, 2, 1, 7$. Thus $\frac{564}{85} = 6 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{7}$ with $t = 6$ and $q_6 = 7$, which is answer **(b)**.

5. The sequence of continued fraction convergents of a real number includes the following three consecutive terms:

$$\frac{120}{7}, \quad \frac{A}{B}, \quad \frac{874}{51}$$

where A, B are coprime positive integers. The value of A/B is

(a) 97/5 (b) 127/7 (c) 377/22 (d) none of these.

Solution: There are several ways of seeing this, using combinations of formulas from Section 2 of Workbook 4. The simplest is this: let the three convergents be $A_{n-1}/B_{n-1}, A_n/B_n, A_{n+1}/B_{n+1}$. Then from $A_{n+1} = q_{n+1}A_n + A_{n-1}$ and $B_{n+1} = q_{n+1}B_n + B_{n-1}$ we get $A_n/B_n = (A_{n+1} - A_{n-1})/(B_{n+1} - B_{n-1}) = (874 - 120)/(51 - 7) = 754/44 = 377/22$: answer **(c)**. (One real number for which these convergents do occur is 17.137.)

6. Which of the following quadratic equations has a *reduced* quadratic irrational root?

(a) $x^2 + 7x + 12 = 0$ (b) $x^2 - 10x + 7 = 0$ (c) $x^2 - 3x - 1 = 0$ (d) $x^2 + 2x - 1 = 0$

Solution: You can use the quadratic formula to find the roots in each case, thus eliminating **(a)**(which has rational roots $-4, -3$), **(b)**(whose roots are $5 \pm 2\sqrt{3}$, neither of which is < 0) and **(d)**(whose roots are $(-1 \pm \sqrt{2})$, neither of which is > 1). Alternatively if α is a reduced root of $x^2 + bx + c$, with conjugate α' then $\alpha + \alpha' > 0 \implies b < 0$ and $\alpha\alpha' < 0 \implies c < 0$, which only leaves **(c)**.

7. If α is the quadratic irrational represented by the periodic continued fraction

$$2, \overline{3,\, 1} = 2 + \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\cdots,$$

then

(a) $\alpha = (9 - \sqrt{21})/6$ (b) $3\alpha^2 - 9\alpha - 5 = 0$ (c) $2.5 < \alpha < 2.6$ (d) none of these

Solution: We have $\alpha - 1 = 1 + \frac{1}{3+}\,\frac{1}{\alpha - 1} = 1 + \frac{\alpha-1}{3\alpha-2} = \frac{4\alpha-3}{3\alpha-2}$, so α satisfies $3\alpha^2 - 9\alpha + 5 = 0$, giving $\alpha = 3/2 + \sqrt{21}/6$. This rules out **(a)** and **(b)**. To check **(c)**: $2.5 < \alpha < 2.6 \iff 1 < \sqrt{21}/6 < 1.1$, which is false since $1^2 = 1 \not< \frac{21}{36}$. So **(d)** is the correct answer.

8. The continued fraction $\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{2}$ represents the rational number

(a) 11/42 (b) 15/36 (c) 42/11 (d) 36/15

Solution: Using the quotients $0, 3, 1, 4, 2$ and the recurrence formulae we find that the numerator sequence A_n is $0, 1, 1, 5, 11$ and the denominator sequence B_n is $1, 3, 4, 19, 42$. Thus the number is $\frac{11}{42}$ which is **(a)**. (It is easy to get the fraction upside down by mistake: but it should be obvious from the CF that the number is less than 1.)

9. The continued fraction expansion of $n + \sqrt{61}$ is purely periodic when

(a) $n = -6$ (b) $n = 7$ (c) $n = -7$ (d) $n = 0$

Solution: We require $\alpha = n + \sqrt{61} > 1$ and $-1 < \alpha' < 0$ where $\alpha' = n - \sqrt{61}$. So $n < \sqrt{61} < n + 1$ and $n = [\sqrt{61}] = 7$: answer **(b)**.

10. The polynomial $[q_0, q_1, q_2, q_3]$ evaluated at $q_0 = 4$, $q_1 = 3$, $q_2 = 2$ and $q_3 = 1$ is equal to

(a) 7 (b) 30 (c) 13 (d) 43

Solution: Using the recurrence formula starting with $0, 1$ and with the quotient sequence $4, 3, 2, 1$ gives in turn $0, 1, 4, 13, 30, 43$: so the answer is **(d)**.

11. The irrational number α has continued fraction expansion $q_0 + \frac{1}{q_1+}\,\frac{1}{q_2+}\cdots$.

Given that $[60\alpha] = 93$, then q_1 is

(a) 1 (b) 2 (c) 3 (d) none of these.

Solution: $[60\alpha] = 93 \implies 93 < 60\alpha < 94 \implies \frac{31}{20} < \alpha < \frac{47}{30}$. (Both inequalities are strict since α is irrational.) Since $31/20 = 1 + 11/20 < 2$, $q_0 = [\alpha] = 1$. Now (subtracting 1 throughout the inequality), $\frac{11}{20} < \alpha - 1 < \frac{17}{30}$. Then

$$1 < \frac{30}{17} < \frac{1}{\alpha - 1} < \frac{20}{11} < 2,$$

so $q_1 = [1/(\alpha - 1)] = 1$, which is answer **(a)**.