

Test 2 Solutions

1. Let $1 \leq r, s \leq 319$, regarded as elements of $\mathbb{Z}/319\mathbb{Z}$ (the ring of integers modulo 319). Then
- (a) there exists an r such that $rs \notin \{0, 1\}$ for all s .
 - (b) for all r there exists an s with $rs = 0$.
 - (c) there exists an r such that $rs \neq 1$ for all s .
 - (d) for all r and s the product rs is non-zero.

Solution: **(a)** is false, for if r is coprime to 319 then there is an s with $rs = 1$, and otherwise there is an s with $rs = 0$. **(b)** is false ($r = 1$ fails). **(c)** would be true if and only if 319 were not prime (since it asserts that not every r is invertible), while **(d)** would be true if and only if 319 were prime (since it asserts that there are no zero-divisors). So to decide, we do need to know whether or not 319 is prime. In fact, $319 = 11 \times 29$, and so the answer is **(c)**.

2. Let x be the unique integer in $\{1, \dots, 500\}$ satisfying $x \equiv 2 \pmod{15}$ and $x \equiv 11 \pmod{29}$; then $x \equiv a \pmod{10}$ for
- (a) $a = 0$
 - (b) $a = 2$
 - (c) $a = 5$
 - (d) $a = 7$

Solution: The solution to this Chinese Remainder problem is $x \equiv 272 \pmod{435}$, from $\text{hcf}(15, 29) = 1 = 30 - 29$, $x \equiv 30 \cdot 11 - 29 \cdot 2 = 330 - 58 = 272 \pmod{435}$, where $\text{lcm}(15, 29) = 15 \cdot 29 = 435$. Since $1 \leq x \leq 400$, in fact $x = 272$, so $x \equiv 2 \pmod{10}$ and the answer is **(b)**.

3. The number of natural numbers n such that $\varphi(\varphi(n))$ is odd is equal to
- (a) 4
 - (b) 5
 - (c) 6
 - (d) 7

Solution: Let $m = \varphi(n)$; observe that $\varphi(m)$ is odd if and only if $m \leq 2$, and hence we are looking for values of n such that $\varphi(n)$ is either 1 or 2. The equality $\varphi(n) = 1$ holds if and only if $n \in \{1, 2\}$; the equality $\varphi(n) = 2$ holds if and only if $n \in \{3, 4, 6\}$. Thus there are 5 values and the correct answer is **(b)**.

4. The number of integers a with $0 \leq a \leq 101$ such that the simultaneous congruences
- $$\begin{cases} x \equiv a \pmod{102} \\ x \equiv 7 \pmod{2013} \end{cases} \text{ have a solution is}$$
- (a) 1
 - (b) 6
 - (c) 34
 - (d) 51

Solution: Since $\text{hcf}(102, 2013) = 3$, a solution exists if and only if $a \equiv 7 \equiv 1 \pmod{3}$. In the range $0 \leq a \leq 101$ there are 34 solutions, $1 + 3k$ for $k = 0, 1, 2, \dots, 33$: answer **(c)**.

5. Let $S = \{1, 3, 7, 15, 32, \dots\}$ denote the set of natural numbers of the form $2^n - 1$. Then
- (a) there is a prime $p > 2$ which divides no member of S .
 - (b) there is a prime p which divides exactly one member of S .
 - (c) all primes $p > 2$ divide infinitely many members of S .
 - (d) there is only one prime that divides infinitely many members of S .

Solution: For all primes p other than 2, let d be the order of 2 modulo p ; then $d \mid n \implies 2^n \equiv 1 \pmod{p} \implies p \mid 2^n - 1$. Hence all primes $p > 2$ divide infinitely many terms of the sequence, while 2 divides none of them: answer **(c)**.

6. Let a and n be natural numbers. Then $\varphi(2^a n) = 2^a \varphi(n)$ if and only if
- (a) n is even.
 - (b) n is odd.
 - (c) a is even
 - (d) a is odd.

Solution: (b) and (d) are wrong, since for example $1 = \varphi(2 \cdot 1) \neq 2\varphi(1) = 2$. (c) is wrong, since for example $2 = \varphi(2^2 \cdot 1) \neq 2^2\varphi(1) = 4$. The only possibility left is (a). Alternatively, write $n = 2^b \cdot n'$, with n' odd; it follows from the multiplicativity of φ that $\varphi(2^a n) = \varphi(2^{a+b} n') = 2^{a+b-1} \varphi(n') = 2^a \varphi(2^b) \varphi(n') = 2^a \varphi(2^b n') = 2^a \varphi(n)$, so that again (a) is true.

7. The unit group \mathbb{U}_{15} of units of $\mathbb{Z}/15\mathbb{Z}$
- (a) has at least one element of order 3.
 - (b) has no element of order 2.
 - (c) is cyclic of order 8.
 - (d) has at least one element of order 4.

Solution: Since $\varphi(15) = 8$ the group has order 8, consisting of $\{1, 2, 4, 7, 8, 11, 13, 14\} \pmod{15}$, or $\{\pm 1, \pm 2, \pm 4, \pm 7\} \pmod{15}$. Since $(\pm 2)^2 \equiv (\pm 7)^2 \equiv 4 \pmod{15}$ and $(\pm 1)^2 \equiv (\pm 4)^2 \equiv 1 \pmod{15}$, the group contains no element of odd order, no elements of order 8, and it contains three elements of order 2 and four elements of order 4: answer (d).

8. Let n and j be a natural numbers. The equality $\varphi(n^j) = n^{j-1} \varphi(n)$ holds
- (a) for all choices of n, j .
 - (b) only if n is even.
 - (c) only if $j = 1$.
 - (d) only if n is prime.

Solution: Observe that n and n^j have the same prime factors; thus we have $\varphi(n^j) = n^j \prod_{p|n^j} (1 - \frac{1}{p}) = n^{j-1} n \prod_{p|n} (1 - \frac{1}{p}) = n^{j-1} \varphi(n)$, and the answer is (a).

9. As a function from the set $\mathbb{N} = \{1, 2, 3, \dots\}$ of natural numbers to itself, the Euler phi-function is
- (a) always even
 - (b) always odd
 - (c) odd only once
 - (d) none of these

Solution: The Euler-phi function is certainly odd at least twice, for instance $\varphi(1) = \varphi(2) = 1$; it is also even, for instance $\varphi(4) = 2$: (d).

10. The number of integers n in the range $30 \leq n \leq 40$ for which the map

$$f : \mathbb{Z}/300n\mathbb{Z} \rightarrow \mathbb{Z}/300\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$$

defined by

$$x \pmod{300n} \mapsto (x \pmod{300}, x \pmod{n})$$

is a bijection is

- (a) 0
- (b) 2
- (c) 4
- (d) 6

Solution: The map is a bijection if and only if $\text{hcf}(300, n) = 1$, which is if and only if n is divisible by neither 2, 3 nor 5. So $n = 31, 37$ and the answer is (b).

11. Let $n \in \mathbb{N}$ be even and let p be a prime factor of the number $n^{64} - 1$. The order of n modulo p is
- (a) $p - 1$
 - (b) a multiple of 64
 - (c) a divisor of 64
 - (d) $\varphi(n)$

Solution: $p \neq 2$ since $n^{64} - 1$ is odd. Now $n^{64} \equiv 1$ and therefore 64 is divisible by the order of n modulo p . Answer (c).