MA246 Number Theory 2012

# Test 1 Solutions

**1.** Let $n \in \mathbb{N}$. The number of distinct sets of the form $n\mathbb{Z} + m$ as $m$ runs through $\mathbb{Z}$ is

(a) infinite. (b) equal to $n-1$. (c) always greater than 1. (d) equal to $n$.

*Solution:* These sets are the distinct residue classes modulo $n$, and there are exactly $n$ of them (obtained by taking $m = 0, 1, \ldots, n-1$, for example). The answer is **(d)**.

**2.** Let $b \in \mathbb{N}$. The congruence $5x \equiv 2b \pmod{70}$

(a) is satisfied by infinitely many integer values of $x$ whatever the value of $b$.
(b) has solutions in $x$ for some values of $b$, but not for others.
(c) has exactly one congruence class of solutions in $x$ for every even value of $b$.
(d) has infinitely many solutions in $x$ only when $b = 0$.

*Solution:* There are no solutions when $b$ is not divisible by 5 since $h = \text{hcf}(5, 70) = 5$ must divide $b$. There are always solutions when $b$ is divisible by 5, and there are then exactly $h = 5$ (congruence classes of) solutions. So the answer is **(b)**.

**3.** On May 3rd 1952 the first plane landed on the North Pole; that day was a:

(a) Thursday.
(b) Friday.
(c) Saturday.
(d) Sunday.

*Solution:* We need to take any date for which the day of the week is already known as a base-point. I will take the day of the test, namely Friday 3/5/2013. You may have chosen a different one.
From 3/5/1952 to 3/5/2013 was 61 years and one day, counting $365 \equiv 1 \pmod{7}$ each, except that 14 of these were leap years and count $366 \equiv 2 \pmod{7}$. So the number of days until 3/5/2013 was $\equiv 61 + 14 + 1 = 76 \equiv 6 \pmod{7}$. So in all the number of days from the day the first plane landed on the North Pole until 3/5/2013 is $\equiv 6 \pmod{7}$. Today is Friday 3/5/2013, so counting back 6 days (or forward 1 day) gives Saturday, and the answer is **(c)**.

**4.** Fix $n \in \mathbb{N}$ and let $a, b \in \mathbb{Z}$. The map $x \mapsto ax + b \pmod{n}$ from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$ is a bijection

(a) for all values of $a$ and $b$.
(b) for some values of $a$ and all values of $b$.
(c) for all values of $a$ and some values of $b$.
(d) for no values of $a$ and $b$.

*Solution:* The map $x \mapsto ax$ is a bijection if and only if $\text{hcf}(a, n) = 1$ , and the map $x \mapsto x + b$ is always a bijection, so the composite map $x \mapsto ax + b$ is a bijection iff $\text{hcf}(a, n) = 1$. [Alternatively, the congruence $ax + b \equiv y \pmod{n}$ has a unique solution for all $x$ iff $a$ and $n$ are coprime.] So the answer is **(b)**.

**5.** A frisbee is consistently thrown six to the right in a ring of 15 players. The number of players who never touch the frisbee is

(a) 3. (b) 5. (c) 10. (d) 15.

*Solution:* Since the highest common factor of 6 and 15 is 3, every third player will eventually touch the frisbee, so $5 = 15/3$ will touch it and $15 - 5 = 10$ will not: the answer is **(c)**.

**6.** Which of the following sets forms a complete set of residues modulo 5:

(a) $\{0, -1, 2, -3, 4\}$.
(b) $\{2^0, 2^1, 2^2, 2^3, 2^4\}$.
(c) $\{1^2, 2^2, 3^2, 4^2, 5^2\}$.
(d) $\{176, 187, 198, 209, 110\}$.

*Solution:* A set of 5 integers will form a complete set of residues modulo 5 if and only if there are no repeats. This rules out **(a)**, since $-1 \equiv 4$, **(b)**, since $2^4 \equiv 2^0 \equiv 1 \pmod 5$ and **(c)** since $1^2 \equiv 4^2$. The answer is therefore **(d)**. Alternatively, you can easily observe that the integers in set **(d)** are congruent to $1, 2, 3, 4, 0$ so form a complete set.

**7.** Let $r \in 10\mathbb{Z} + 7^9$ with $1 \leq r \leq 9$. Then

(a) $r = 1$ (b) there is no such $r$ (c) $r = 7$ (d) $r = 9$.

*Solution:* Modulo 10 we have $7^2 = 9 \equiv -1$ so $7^8 \equiv (-1)^4 \equiv 1$, and $r \equiv 7^9 \equiv 7 \pmod{10}$; so $r = 7$, the answer is **(c)**.

**8.** The Linear Diophantine Equation $123x + 456y = 2013$ has

(a) infinitely many
(b) exactly one
(c) more than 1 but a finite number of
(d) no
solution(s) in integers $x, y$.

*Solution:* hcf$(123, 456) = 3$ which divides 2013, so there are infinitely many solutions: **(a)**.

**9.** The number of integers $b$ with $0 \leq b \leq 20$ for which the congruence $33x \equiv b \pmod{21}$ has exactly **three** integer solutions $x$ with $0 \leq x \leq 20$ is

(a) 1 (b) 3 (c) 5 (d) 7

*Solution:* hcf$(33, 21) = 3$. If $3 \mid b$ then there are 3 solutions (modulo 21), otherwise there are none. So there are 3 solutions for $b = 0, 3, 6, 9, 12, 15, 18$ and the answer is **(d)**.

**10.** A solution to the congruence: $810x \equiv 198 \pmod{801}$ is

(a) $x = -1$ (b) $x = 9$ (c) $x = 111$ (d) No solution exists.

*Solution:* First simplify the congruence to $9x \equiv 198 \pmod{801}$ (since $810 - 801 = 9$). Now divide by 9 to get the equivalent congruence $x \equiv 22 \pmod{89}$. Clearly **(c)** satisfies this.

**11.** Suppose that $m \equiv 3 \pmod{11}$ and that $10^6 \leq m \leq 10^6 + 10$. Then $m$ is equal to

(a) 1000002 (b) 1000004 (c) 1000006 (d) 1000008.

*Solution:* Modulo 11 we have $10^6 \equiv (-1)^6 \equiv 1$, so $m = 10^6 + 2$ and the answer is **(a)**.