Solutions for Test 1

NOTE

1. These solutions are not necessarily in the order of your particular test paper. It should be straightforward matching your question order with the order of the paper you answered.

2. If you got a question wrong, understand why you got it wrong.

3. If you're not sure about a question, discuss it in your supervision group or with your tutor. If you're still not sure, have a word with your lecturer.

1. How many of the following numbers are divisible by *at least one* prime number: 72373, 399421, 39161 (a) None of them; (b) One of them; (c) Two of them; (d) Three of them.

answer:(d) Every natural number larger than 1 is divisible by a prime.

2. Which of the following is a true statement? (a) There is a prime number which is larger than any other prime number. (b) There is a smallest element of the set of natural numbers x with the property that $x^3 + 5x^2 > 100$. (c) If natural numbers $p_1, ..., p_n$ are prime then $1 + (p_1 p_2 ... p_n)$ is also a prime number; (d) none of the above.

answer:(b) The fact that (a) is false was one of the steps in proving that there are infinitely many prime numbers. By the Well Ordering Principle, to establish that (b) is true we need only show that there is one such x. Trying $x = 100$ works, so it is true. (c) wasn't proved in the notes and isn't true; the number $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 + 1$ is a multiple of 19, for example. We didn't expect you to find the counterexample, only to know that it wasn't proved, and (b) is true.

3. Suppose that $P(n)$ is a statement, for each integer $n \in \mathbb{Z}$. Suppose also that $P(2)$ is true, and that if $P(n)$ is true then so also is $P(n + 2)$. Consider the following statements:

i) $P(100)$ is true;
ii) $P(n)$ is true for all integers $n \geq 2$;
iii) $P(2n)$ is true for all integers $n \geq 1$;
iv) $P(2n)$ is true for all $n \in \mathbb{Z}$.

The number of (i)-(iv) which must be true is: (a) 1; (b) 2; (c) 3; (d) 4.

answer:(b) Part iii) is true. Here is one proof of part iii). Let $T \subset \mathbb{N}$ be the set of numbers n greater than or equal to 1 such that $P(2n)$ is false. If T is not empty then let t be the smallest element (assuming the well-ordering principle). We know $2t \neq 2$ since $P(2)$ is actually true. Then $t - 1$ is greater than or equal to 1. If $P(2(t - 1))$ were true $P(2t)$ would be true. Then $t - 1 \in T$ and this is not possible. Another proof of part iii) is to use the Principle of Induction slightly modified to start with the value $t = 1$. Then part i) is of course also true. The other parts are not true, for example if $P(n)$ is the statement "n is even and larger than 0" then ii) and iv) are false.

4. Let $n \geq 2$ be a natural number and let $S \subset \mathbb{N}$ be the set whose elements are divisors of n. Consider the following statements: (i) S is bounded above; (ii) S has a greatest element; (iii) S has a least element (iv) S contains a prime number. How many of these statements are true? (a) one. (b) two. (c) three. (d) four.

answer: (d) They are all true.

5. Consider the following subsets of the set $\mathbb{N}$ of natural numbers i) The set of prime numbers; ii) The set of even numbers; iii) The set of non-prime numbers. How many of these sets are bounded below? a) 0; b) 1; c) 2; d) 3.

answer:(d) Every set of natural numbers is bounded below.

6. What is the remainder when 25 is divided by 32 (in the sense of intger division with remainder)? a) It is undefined because $25 < 32$; b) 0; c) 7; d) 25.

answer:(d) Division with remainder is well-defined providing the number you're dividing by is non-zero: here, $32 \neq 0$, so we can divide and we get a remainder of 25.

7. Consider the equations $m = 9987n + 3989; m = 7871n + 723$. For how many integer values of n can they both be true? a) 0 b) 1 c) 2 d) 3

answer:(a) They cannot both be true. Suppose they were both true: then n would have to solve $9987n + 3989 = 7871n + 723$, which rearranges to $2116n = -3266$, which clearly has no integer solutions.

8. Consider the integers (i) 1001001; (ii) 1001001001; (iii) 110010010011; (iv) 1002003. How many of them are divisible (without remainder) by 3? Answer (a) 1; (b) 2; (c) 3; (d) 4.

answer:(c) You can do this the long way, or you can recall that an integer (base 10) is divisible by 3 if and only if the sum of the digits is divisible by 3.

9. Consider the expression

$$\sum_{k=1}^{n} k^2 .$$

Compare this with the six expressions:

$$\sum_{k=0}^{n-1}(k+1)^2 ; \quad \sum_{k=0}^{n-1}(k-1)^2 ; \quad \sum_{j=1}^{n} j^2 ; \quad \sum_{k=2}^{n+1}(k-1)^2 ; \quad \sum_{j=1}^{n} k^2 ; \quad n^2 + \sum_{k=1}^{n-1} k^2 .$$

How many of these six expressions have the same value as the one above: (a) 1; (b) 2; (c) 3; (d) 4.

Answer: (d). This question is to check your understanding of the the summation sign and the use of indices. The first, third, fourth and sixth expressions mean the same thing as the original one.

10. The sum $1^2+2^2+\ldots n^2$ is equal to (a) $\frac{1}{6}n(2n+1)(2n+2)$; (b) $\frac{1}{6}n(n+1)(2n+1)$; (c) $\frac{1}{6}n(n+1)(n+2)$; (d) $\frac{1}{3}n^2(2n+1)$.

answer:(b) The proof of this was an exercise; even if you haven't done it, testing with $n = 2$ quickly rules out the others.

11. The least element of the set $R = \{m - 3qn \in \mathbb{N} : q \in \mathbb{N}\}$, where $m = 555$ and $n = 11$, (a) does not exist (b) is 1 (c) is 5 (d) is 27.

answer: (d) Successively subtracting multiples of 33 from 555 gets you to 27.