Solutions for Test 4

NOTE

1. These solutions are not necessarily in the order of your particular test paper. It should be straightforward matching your question order with the order of the paper you answered.

2. If you got a question wrong, understand why you got it wrong.

3. If you're not sure about a question, discuss it in your supervision group or with your tutor. If you're still not sure, have a word with your lecturer.

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1. Given $n \in \mathbb{N}$, let $\mathscr{D}(n)$ denote the set of natural numbers which divide n. Which of the following statements is true?
(a) $\mathscr{D}(72)$ is in bijection with $\mathscr{D}(108)$; (b) $\mathscr{D}(64)$ is in bijection with $\mathscr{D}(96)$; (c) $\mathscr{D}(48)$ is in bijection with $\mathscr{D}(72)$; (d) $\mathscr{D}(36)$ is in bijection with $\mathscr{D}(48)$.

Answer: (a). One can check that 72 and 108 both have twelve divisors; 64 has seven divisors, but 96 has twelve; 48 has ten divisors, and 36 has nine divisors. More generally, one can note that if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots p_n^{\alpha_n}$ is a prime factorisation (where the p_j are distinct primes), then the number of elements in $\mathscr{D}(n)$ is $(\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_n + 1)$.

2. Define $f\colon \mathbb{Z} \to \mathbb{N}$ by

$$f(x) = \begin{cases} 1 - 2x & \text{if } x \leq -3, \\ x + 3 & \text{if } -3 < x \leq 3, \\ 2x & \text{if } x > 3. \end{cases}$$

Then (a) f is neither injective nor surjective; (b) f is injective but not surjective; (c) f is surjective but not injective; (d) f is both injective and surjective.

Answer: (d). Note that $f(-2) = 1$, $f(-1) = 2$, and so on until $f(3) = 6$; then $f(-3) = 7$, $f(4) = 8$, $f(-4) = 9$, $f(5) = 10$, and so on; in particular, $f(-n) = 2n + 1$ and $f(n) = 2n$ for every natural number $n \geq 3$. f is thus both injective and surjective.

3. Recall that $g\colon B \to A$ is a *right inverse* to $f\colon A \to B$ if $f \circ g = \mathrm{id}_B$. Let $f\colon \mathbb{R} \to [-1, 1]$ be the function $f(x) = \sin x$. Then (a) f does not have a right inverse; (b) f has one right inverse; (c) f has two right inverses; (d) f has infinitely many right inverses.

Answer: (d). It has infinitely many right inverses. Let $\arcsin\colon [-1, 1] \to [-\pi/2, \pi/2]$ be the usual inverse of the restriction of sin to the interval $[-\pi/2, \pi/2]$. Then for each $n \in \mathbb{Z}$, $g_n(y) := \arcsin(y) + 2n\pi$ is a right inverse for sin; so is $h_n(y) := (2n + 1)\pi - \arcsin(y)$. In fact, you can even let n vary with y to get even more possibilities. (Can you describe the set of all possible right inverses?)

4. Let $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$. How many of these functions are *surjective*?
(i) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$; (ii) $g\colon \mathbb{R} \to \mathbb{R}_{>0}$, $g(x) = e^x$; (iii) $h\colon \mathbb{R}_{>0} \to \mathbb{R}$, $h(x) = 1/x$.
(a) None are surjective; (b) One is surjective, and two are not; (c) Two are surjective, and one is not; (d) All three are surjective.

Answer: (b). Just one is surjective, namely g, since e^x is always positive for any $x \in \mathbb{R}$; f is not surjective, as no real number squares to -1; and h is not surjective, as the reciprocal of a positive number must be positive.

5. Let A, B be subsets of some set X. Which of the following statements is always true?
(a) $(A \setminus B) \cap (B \setminus A) = A \cup B$;
(b) $(A \setminus B) \cap (B \setminus A) = A \cap B$;
(c) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$;
(d) $(A \setminus B) \cup (B \setminus A) = (A \cap B) \setminus (A \cup B)$.

Answer (c). One can easily prove that (c) is true using de Morgan's laws. To see the others are false, one need only observe that the left-hand side of (a) and (b) must be empty, while the right-hand side need not be; similarly, the right-hand side of (d) must be empty, while the left-hand side need not be.

6. Let $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$. How many of these functions are *injective*?
(i) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$; (ii) $g\colon \mathbb{R} \to \mathbb{R}_{>0}$, $g(x) = e^x$; (iii) $h\colon \mathbb{R}_{>0} \to \mathbb{R}$, $h(x) = 1/x$.
(a) None are injective; (b) One is injective, and two are not; (c) Two are injective, and one is not; (d) All three are injective.

Answer: (c). Two are injective: f is not injective, as $(-x)^2 = x^2$, but g and h are injective.

7. Let $f\colon A \to B$ and $g\colon B \to C$ be functions such that $g \circ f\colon A \to C$ is a bijection. How many of the following statements are true?
(i) f is necessarily injective; (ii) f is necessarily surjective;
(iii) g is necessarily injective; (iv) g is necessarily surjective.
(a) 1 is true and 3 are false; (b) 2 are true and 2 are false; (c) 3 are true and 1 is false; (d) all four are true.

Answer: (b). Two are true, namely (i) and (iv); and two are false, namely (ii) and (iii). In other words, if $g \circ f$ is bijective, then g is necessarily surjective and f is necessarily injective. However, f need not be surjective, and g need not be injective, as the following example shows: let $A = C = \{0\}$, $B = \{0,1\}$, and define $f(0) = 0$, $g(0) = 0$, $g(1) = 0$. Then f is injective but not surjective, and g is surjective but not injective (and $g \circ f$ is indeed a bijection).

8. Which of these statements is *not* a tautology?
(a) $(P \implies Q) \iff (\neg(P \wedge (\neg Q)))$;
(b) $(P \wedge (P \implies Q)) \implies Q)$;
(c) $(\neg(P \implies Q)) \iff (Q \implies P)$;
(d) $(P \implies Q) \iff ((\neg Q) \implies (\neg P))$.

Answer: (c). One can verify that all but (c) are tautologies using, for example, truth tables. Thinking laterally, if (c) were true, then in particular $Q \implies P$ would imply $\neg(P \implies Q)$, which is by no means always true (for example, it's false if $P \iff Q$).

9. Let S be the set of all students currently registered at the University of Warwick. Define a relation $\sim$ on S by $x \sim y$ if person x has shaken hands with person y (that is, person x has clasped their right hand into the right hand of person y). How many of the following statements are true?
(i) $\sim$ is reflexive; (ii) $\sim$ is symmetric; (iii) $\sim$ is transitive.
(a) all three are false; (b) 1 is true and 2 are false; (c) 2 are true and 1 is false; (d) all three are true.

Answer: (b). Statement (ii) is true, and the others are false: a person cannot shake hands with themselves, so $\sim$ is not reflexive. It is clearly symmetric, since if x has shaken hands with y then y has shaken hands with x. But $\sim$ is not transitive, since if x and y have shaken hands and y and z have shaken hands, it does not necessarily follow that x and z have shaken hands.

10. Let $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$, and $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ by $g(x) = \sqrt{x}$ (that is, the *non-negative* square root of x). Which of the following statements is true?
(a) Neither $f \circ g$ nor $g \circ f$ are well-defined functions;
(b) $f \circ g$ is a well-defined function, but $g \circ f$ is not;
(c) $g \circ f$ is a well-defined function, but $f \circ g$ is not;
(d) Both $f \circ g$ and $g \circ f$ are well-defined functions.

Answer: (b). Since $\sqrt{x}$ is only defined for $x \geq 0$, $\sqrt{\sin x}$ is only defined when $\sin x \geq 0$, i.e. when $x \in [2n\pi, (2n+1)\pi]$ for any integer n. In any case, $\sqrt{\sin x}$ is not defined on all of $\mathbb{R}$, which is the domain of f, and hence $g \circ f$ is not well-defined. On the other hand, $\sin x$ is defined for any real number x, so in particular $\sin \sqrt{x}$ is always well-defined, and hence $f \circ g$ is well-defined.

11. Let A, B, C, D be subsets of a set X. Which of the following statements is always true?
(a) $(A \cap B) \cup (A \setminus (B \cap (A \cup C \cup D))) = A$;
(b) $(A \cap B) \cup (A \setminus (B \cap (A \cup C \cup D))) = B$;
(c) $(A \cap B) \cup (A \setminus (B \cap (A \cup C \cup D))) = C$;
(d) $(A \cap B) \cup (A \setminus (B \cap (A \cup C \cup D))) = D$.

Answer: (a). Notice that $A \setminus (B \cap (A \cup C \cup D)) = (A \setminus B) \cup (A \setminus (A \cup C \cup D)) = A \setminus B$, so we get that $(A \cap B) \cup (A \setminus (B \cap (A \cup C \cup D))) = (A \cap B) \cup (A \setminus B) = A$. The others can be disproved by easy counterexamples (or because this one is right and you're told to choose one right answer!)