Solutions for Test 3

NOTE

1. These solutions are not necessarily in the order of your particular test paper. It should be straightforward matching your question order with the order of the paper you answered.

2. If you got a question wrong, understand why you got it wrong.

3. If you're not sure about a question, discuss it in your supervision group or with your tutor. If you're still not sure, have a word with your lecturer.

1. How many of the following numbers have a decimal expansion which is eventually constant (that is, the decimal expansion is eventually a single digit repeating forever)?

$$\frac{5}{11}; \quad \frac{49}{90}; \quad \frac{1}{15}+\frac{\sqrt{3}}{2}; \quad \frac{14}{25}.$$

(a) 0; (b) 1; (c) 2; (d) 3.

Answer: (c). Firstly, $\frac{1}{15}+\frac{\sqrt{3}}{2}$ is clearly irrational, so its decimal expansion cannot be eventually constant (it isn't even periodic). Of the three fractions, note that $\frac{5}{11} = 0.4545\overline{45}$, which isn't eventually constant, but $\frac{49}{90} = 0.54444\overline{4}$ and $\frac{14}{25} = 0.56$, which are.

2. Consider the quadratic equation $x^2 - 2x + 5 = 0$. Which of the following statements is true?
(a) The solutions of the quadratic equation are integers.
(b) The solutions of the quadratic equation are rational numbers, but are not integers.
(c) The solutions of the quadratic equation are real numbers, but are not rational.
(d) The solutions of the quadratic equation are complex numbers, but are not real.

Answer: (d). The solutions are $1 \pm 2i$, as found by the quadratic formula or by factorising; these are complex, but not real.

3. Consider the solutions of the cubic equation $x^3 - 2 = 0$ over the complex numbers. How many of the following statements are true?
(i) All three solutions are real.
(ii) At least one of the solutions is a positive irrational number.
(iii) The three solutions add to zero.
(iv) At least one of the solutions is a positive rational number.
(v) At least one of the solutions is an integer.
(vi) At least one of the solutions is complex but not real.
(a) 1 is true and 5 are false; (b) 2 are true and 4 are false; (c) 3 are true and 3 are false; (d) 4 are true and 2 are false.

Answer: (c). Three are true, namely (ii), (iii) and (vi), and three are false, namely (i), (iv) and (v). The three solutions are $\sqrt[3]{2}$, which is positive and irrational, and $\sqrt[3]{2}\left(\frac{-1\pm i\sqrt{3}}{2}\right)$, which are complex but not real. None are integers or rational numbers. One can check by direct computation that the three roots add to zero, but this follows from them all being $\sqrt[3]{2}$ times the cube roots of unity.

4. Let x be the complex number which solves $(1+i)x = (2i-1)\overline{(2+i)}$. The argument of x equals: (a) $\pi/6$; (b) $\pi/4$; (c) $\pi/3$; (d) $\pi/2$.

Answer: (b). Simple manipulation shows that $x = \frac{5i}{1+i} = \frac{1}{2}(5+5i)$, so the argument of x is $\arctan\frac{5/2}{5/2} = \pi/4$.

5. What is $1 + \cos(\pi/5) + \cos(2\pi/5) + \cos(3\pi/5) + \cos(4\pi/5)$ equal to? (a) 1; (b) 1/2; (c) 0; (d) $\frac{1}{1-\cos(\pi/5)}$.

Answer (a). By an almost identical method to example 5.27 in the lecture notes, the sum is equal to 1.

6. Which of the following complex numbers has the largest imaginary part?
(a) $8(\cos(\pi/2) + i\sin(\pi/2))$; (b) $2i + e^{i\pi/2}$; (c) $\sqrt{3} + 5i$; (d) $7e^{i\pi/3}$.

Answer: (a). The imaginary parts are as follows: (a) $\mathrm{Im}(8(\cos(\pi/2) + i\sin(\pi/2))) = 8$; (b) $\mathrm{Im}(2i + e^{i\pi/2}) = \mathrm{Im}(2i + i) = 3$; (c) $\mathrm{Im}(\sqrt{3} + 5i) = 5$; and (d) $\mathrm{Im}(7e^{i\pi/3}) = \mathrm{Im}(\frac{7}{2}(1 + \sqrt{3}i)) = \frac{7\sqrt{3}}{2} < 7$, as $\sqrt{3} < 2$; from which it is clear that (a) is the largest.

7. Recall that the *period* of a decimal expansion of a rational number is the length of the cycle of repeating digits in its decimal expansion: for example $1/6 = 0.16666\overline{6}$ has period 1, while $1/7 = 0.\overline{142857}$ has period 6. Which of the following statements is true for all prime numbers $p > 5$?
(a) The decimal expansion of $1/p$ always has period $p - 1$.
(b) The decimal expansion of $1/p$ always has period strictly less than $p - 1$.
(c) The period of the decimal expansion of $1/p$ must divide $p - 1$.
(d) The period of the decimal expansion of $1/p$ must divide p.

Answer: (c). The examples $1/7 = 0.\overline{142857}$ with period 6 and $1/11 = 0.0909\overline{09}$ with period $2 \neq 10$ show that (a), (b) and (d) must be false, leaving only (c). To prove (c), one needs an argument along the lines of theorem 4.5.

8. Let $n \geq 2$ be a natural number, and let $\zeta = e^{2\pi i/n}$ be a complex n^{th} root of unity. Consider the product $P = 1 \cdot \zeta \cdot \zeta^2 \cdot \dots \cdot \zeta^{n-1}$. Which of the following statements is true?
(a) The product P is always equal to 1.
(b) The product P is always equal to -1.
(c) The product P equals 1 when n is odd, and -1 when n is even.
(d) The product P equals 1 when n is even, and -1 when n is odd.

Answer: (c). By noting that $1 + 2 + \cdots + (n-1) = n(n-1)/2$, we see that $P = \zeta^{n(n-1)/2}$. Now, $\zeta^{n/2} = e^{\pi i} = -1$, so $P = (-1)^{n-1}$, which equals 1 when n is odd, and -1 when n is even.

9. Let a and b be real numbers such that $a - b$ and ab are both rational numbers. Which of the following is necessarily true:
(a) a/b is rational; (b) $a^2 + b^2$ is rational; (c) $a + b$ is rational; (d) a and b are both rational.
[Hint: If you have not done a question exactly like this you will have to think.]

Answer: (b). $a^2 + b^2 = (a-b)^2 + 2ab$ which is rational, as $a + b$ and ab are rational. You can rule out all the others: $a = b = \sqrt{2}$ is a counterexample for (c) and statements, and $a = -1 + \sqrt{2}$, $b = 1 + \sqrt{2}$ is a counterexample to (a) and (d).

10. Which of the following complex numbers has the largest absolute value?
(a) $4e^{3i\pi/2}$; (b) $2 - 4i$; (c) $2i(3 + i)$; (d) $5(\cos(\pi/6) + i\sin(\pi/6))$.

Answer (d). The absolute values are as follows: (a) $|4e^{3i\pi/2}| = 4$; (b) $|2 - 4i| = \sqrt{2^2 + 4^2} = \sqrt{20}$; (c) $|2i(3 + i)| = |-2 + 3i| = \sqrt{2^2 + 3^2} = \sqrt{13}$; (d) $|5(\cos(\pi/6) + i\sin(\pi/6))| = 5$; from which we observe that $\sqrt{13} < 4 < \sqrt{20} < 5$.

11. Which of the following statements is true for *all* complex numbers α and β?
(a) $\mathrm{Re}(\alpha - \beta) = \mathrm{Re}(\alpha) - \mathrm{Re}(\beta)$; (b) $\mathrm{Im}(\alpha\beta) = \mathrm{Im}(\alpha)\mathrm{Im}(\beta)$; (c) $|\alpha + \beta| = |\alpha| + |\beta|$; (d) $|\alpha - \beta| = |\alpha| - |\beta|$.

Answer (a). Writing $\alpha = a + bi$ and $\beta = c + di$, $\mathrm{Re}(\alpha - \beta) = a - c = \mathrm{Re}(\alpha) - \mathrm{Re}(\beta)$, so (a) is true. To see (b) is false, one can take $\alpha = \beta = i$, as then $\mathrm{Im}(\alpha\beta) = 0 \neq 1 = \mathrm{Im}(\alpha)\mathrm{Im}(\beta)$. For both (c) and (d) one can take $\alpha = 1$, $\beta = -1$, since then $|\alpha + \beta| = 0 \neq 2 = |\alpha| + |\beta|$, and $|\alpha - \beta| = 2 \neq 0 = |\alpha| - |\beta|$.