

**Assignment 8**

**March 2017**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems 2, 4 and 5 only must be handed in by **3.00 pm on MONDAY 13 MARCH** (Monday of the tenth week of term), or they will not be marked.

1. An  $n \times n$  matrix  $B$  is called symmetric if  $B = B^T$ . For any  $n \times n$  matrix  $A$ , show that both  $AA^T$  and  $A + A^T$  are symmetric matrices. (Hint: Prove that  $(CD)^T = D^T C^T$  if  $C$  and  $D$  are  $n \times n$  matrices.)

2. Let  $A$  be the  $2 \times 2$  matrix  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ .

Find the smallest value of  $n$  such that the matrices  $A^0 = I, A, A^2, \dots, A^n$  are linearly dependent. [2 marks]

Find the set  $\text{Com}(A) = \{B \in M_2 \mid AB = BA\}$ . [2 marks]

Show that  $\text{Com}(A)$  is a vector space and that  $I, A$  is a basis of  $\text{Com}(A)$ . [3 marks]

3. Solve the following sets of linear equations using Cramer's rule.

(i)  $x - y = -6, \quad 2x + y = 9.$

(ii)  $x - y = 1, \quad y + 2z = 0, \quad 3x - y = -1.$

4. Let  $K$  be a subfield of a larger field  $L$ . (For example,  $(K, L)$  could be  $(\mathbb{Q}, \mathbb{R})$  or  $(\mathbb{R}, \mathbb{C})$ .)

(i) Let  $A$  be an  $m \times n$  matrix with entries in  $K$ . Then  $A$  can be regarded either as a matrix in  $K^{m,n}$  or as a matrix in  $L^{m,n}$ . Show that the rank of  $A$  is the same in either case. (Hint: Use Corollary 8.7 of lecture notes.) [4 marks]

(ii) Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  be vectors in  $K^n$ . Show that  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  are linearly independent in  $K^n$  if and only if they are linearly independent in  $L^n$ . (Hint: Use (i).) [2 marks]

5. Let  $A$  be an  $m \times n$  matrix. If  $k \leq m$  and  $l \leq n$ , then a  $k \times l$  matrix  $B$  is said to be a *submatrix* of  $A$ , if  $B$  can be obtained from  $A$  by deleting some set of  $m - k$  rows and  $n - l$  columns of  $A$ .

(For example, if  $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ , then  $(6), \begin{pmatrix} 3 \\ 11 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \\ 9 & 10 & 12 \end{pmatrix}$  are all examples of submatrices of  $A$ .)

Define the *determinantal rank* of  $A$  to be the largest  $k$  for which  $A$  has a  $k \times k$  submatrix with nonzero determinant. Show that this is equal to the usual row or column rank of  $A$ . [7 marks]

6. Show directly (without using row and column operations on matrices), that if  $T : U \rightarrow V$  is a linear map, then there exist bases of  $U$  and  $V$  such that the matrix of  $T$  looks like

$$\begin{pmatrix} I_r & \mathbf{0}_{r,n-r} \\ \mathbf{0}_{m-r,r} & \mathbf{0}_{m-r,n-r} \end{pmatrix}.$$

(Hint: Choose a basis of the kernel of  $T$ , extend it to a basis of  $U$ , reorder the resulting basis of  $U$ , and then use the images of this basis under  $T$  as part of a basis of  $V$ .)