MA106 – Linear Algebra

Assignment 7 **March 2017**

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **1, 2, 4** *and* **5** *only must be handed in by* **3.00 pm** *on* **MONDAY 6 MARCH** *(Monday of the ninth week of term), or they will not be marked.*

1. Write the following permutations of the set $\{1,2,3,4,5,6,7,8\}$ in cyclic form, then express them as composites of transpositions, and hence decide whether they are even or odd permutations. [1 mark each; marks given for correct answers only]

$$\text{(i)}\ \begin{matrix}1&2&3&4&5&6&7&8\\ \downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\ 8&1&7&5&4&3&2&6\end{matrix};\quad \text{(ii)}\ \begin{matrix}1&2&3&4&5&6&7&8\\ \downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\ 1&3&5&2&4&7&6&8\end{matrix};\quad \text{(iii)}\ \begin{matrix}1&2&3&4&5&6&7&8\\ \downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\ 4&6&5&1&7&2&8&3\end{matrix}.$$

2. Evaluate the following determinants. You may want to use elementary row and/or column operations to reduce the matrix to a simpler form first.

[2 marks for (i), 3 marks for (ii).]

$$\text{(i)}\ \begin{vmatrix}-1&4&-1\\1&1&1\\3&-4&2\end{vmatrix};\qquad \text{(ii)}\ \begin{vmatrix}-1&2&0&5\\2&2&3&0\\1&-1&0&-2\\0&-2&1&-1\end{vmatrix}.$$

3. Prove directly, that if A and B are 2×2 matrices, then $\det(AB) = \det(A)\det(B)$.

4. Let K be the finite field with only the two elements 0 and 1, where $1+1=0$.

(i) How many 2×2 matrices with entries in K are there? [2 marks]

(ii) How many of these are non-singular? [4 marks]

5. Let $n \geq 2$ be an integer. Define A_n to be the matrix whose diagonal entries equal 2, whose entries just above and just below the main diagonal equal -1 and whose remaining entries are 0. Define C_n to be the matrix with a 1 in entries $(1,2), (2,3), \ldots, (n-1,n)$ and $(n,1)$ and with 0 elsewhere. Show that [4+2 marks]

$$\det(A_n) = \underbrace{\begin{vmatrix}2&-1&0&\cdots&0\\-1&2&-1&\ddots&\vdots\\0&-1&2&\ddots&0\\\vdots&\ddots&\ddots&\ddots&-1\\0&\cdots&0&-1&2\end{vmatrix}}_{n\text{ columns}} = n+1 \quad\text{and}\quad \det(C_n) = \underbrace{\begin{vmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&\vdots\\\vdots&\vdots&\vdots&\cdots&0\\0&0&0&\cdots&1\\1&0&0&\cdots&0\end{vmatrix}}_{n\text{ columns}} = (-1)^{n+1}.$$

[*Suggestion*: For A_n, do it for $n=2$ and 3 and then try to use induction on n.]

6. (Continuation of Question 4.)

(i) Find a formula for the number of nonsingular $n\times n$ matrices with entries in the field K of order 2.

[*Hint*: Let the rows of the matrix A be $\mathbf{r}_1, \ldots, \mathbf{r}_n$. Then A is nonsingular if and only if the $\mathbf{r}_i$ are linearly independent, which is the case if and only if, for each i, $\mathbf{r}_i$ is not in the subspace of the rowspace spanned by $\mathbf{r}_1, \ldots, \mathbf{r}_{i-1}$. Use this to count the number of possibilities for $\mathbf{r}_i$, once $\mathbf{r}_1, \ldots, \mathbf{r}_{i-1}$ have been chosen.]

(ii) Using a calculator, show that, if an $n\times n$ matrix over K is chosen at random then, as $n\to\infty$, the probability that the matrix is nonsingular approaches the limit (approximately) $0.288788\ldots$.