

MA106 – Linear Algebra

Assignment 3

February 2017

Answer the questions on your own paper. Write your own name in the top left-hand corner, and your supervisor's name in the top right-hand corner. Solutions to Problems **1**, **2**, **3** and **5** only must be handed in by **3.00 pm** on **MONDAY 6 FEBRUARY** (Monday of the fifth week of term), or they will not be marked.

1. Which of the following subsets of \mathbb{R}^3 are subspaces of \mathbb{R}^3 ? Just answer yes or no in each case. [1 mark for each part]

- (i) $\{(\alpha, \beta, \gamma) \mid \beta^2 = \alpha^2\}$; (ii) $\{(\alpha, \beta, \gamma) \mid \alpha^4 \geq 0\}$;
(iii) $\{(\alpha, \beta, \gamma) \mid \alpha - 2\beta = \gamma\}$; (iv) $\{(\alpha, \beta, \gamma) \mid \alpha + \beta = 3\gamma + 1\}$.

2. Find the dimensions of the subspaces of K^n that are spanned by the following sequences of vectors, and in each case find a subsequence that is a basis of this subspace. [2 marks for each part]

(i) $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 0, 0, -1)$, $(3, 1, 1, -1)$, $(1, 0, 0, 1)$, $(0, 2, 1, 2)$, where $K = \mathbb{R}$;

(ii) $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 0, 0, 1)$, $(0, 1, 0, 1)$, $(1, 1, 1, 1)$, $(0, 1, 1, 0)$, where K is the finite field $\{0, 1\}$ with 2 elements;

(iii) $(-1, 1, 1, -1)$, $(3, -3, -3, 3)$, $(1, -1, -1, 1)$, $(-2, -3, -3, -2)$, $(2, -3, -3, 2)$, $K = \mathbb{R}$;

3. (i) Let W be a subspace of a vector space V . Suppose that $\mathbf{w}_1, \dots, \mathbf{w}_m$ are linearly independent vectors in W which do not span W . Show that there exists $\mathbf{w}_{m+1} \in W$ such that $\mathbf{w}_1, \dots, \mathbf{w}_{m+1}$ are linearly independent. [2 marks]

(ii) Deduce that if W is a subspace of V and V has finite dimension, then W has finite dimension, and $\dim(W) \leq \dim(V)$. [2 marks]

(iii) Let W_1 and W_2 be subspaces of a finite-dimensional vector space V with $W_1 \subseteq W_2$, and suppose that $\dim(W_1) = \dim(W_2)$. Prove that $W_1 = W_2$. [3 marks]

4. Let W be a subspace of a vector space V of finite dimension. Show that W has a complementary subspace in V (i.e. there is a subspace X of V such that $V = W + X$ and $W \cap X = \{\mathbf{0}\}$).

5. Find the ranks and nullities of the following linear maps $T: U \rightarrow V$, and find bases of the kernel and image of T in each case.

(i) $U = \mathbb{R}^4$, $V = \mathbb{R}^4$, $T(\alpha, \beta, \gamma, \delta) = (\alpha - \gamma, -\gamma + \delta, \alpha + \beta, \beta + \delta)$; [2 marks]

(ii) $U = \mathbb{R}[x]_{\leq 2}$ (polynomials of degree at most 2 over \mathbb{R}), $V = \mathbb{R}$, $T(f) = f'(1)$ (first derivative of $f \in U$ evaluated at 1). [1 mark]

6. Describe the following linear maps $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ geometrically. In all cases (x, y) is a vector in \mathbb{R}^2 , and we write $T(x, y)$ rather than $T((x, y))$.

(i) $T(x, y) = (-2x, -2y)$; (ii) $T(x, y) = (0, y)$;

(iii) $T(x, y) = (y, x)$; (iv) $T(x, y) = (x/\sqrt{2} + y/\sqrt{2}, -x/\sqrt{2} + y/\sqrt{2})$.

7. Suppose that the linear map $T: U \rightarrow V$ is a bijection. So T has an inverse map $T^{-1}: V \rightarrow U$. Prove that T^{-1} is a linear map.