Families of conics in the Dwork pencil

Let $\psi \in \mathbb{C}$ be a complex number and let Q_ψ be the quintic hypersurface in $\mathbb{P}^4$ with equation $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \psi x_0 x_1 x_2 x_3 x_4 = 0$. Intersecting Q_ψ with the two-dimensional linear subspace of $\mathbb{P}^4$ determined by

$$\left\{ \begin{array}{lcl} x_0 & = & \alpha x_4 + s \\ x_1 & = & \alpha x_4 - s \\ x_2 & = & \beta x_4 + t \\ x_3 & = & \beta x_4 - t, \end{array} \right.$$

where $\alpha, \beta \in \mathbb{C}$ and s, t are linear forms, we find the plane curve

$$x_4 \Big((1+2\alpha^5+2\beta^5)x_4^4+20\alpha^3x_4^2s^2+20\beta^3x_4^2t^2+10\alpha s^4+10\beta t^4+\psi(\alpha^2\beta^2x_4^4-\beta^2x_4^2s^2-\alpha^2x_4^2t^2+s^2t^2) \Big)$$

in $\mathbb{P}^2_{x_4,s,t}$. The coefficient of x_4 in the previous expression is the evaluation of the conic C_ψ with equation

$$(1+2\alpha^5+\psi\alpha^2\beta^2+2\beta^5)y_0^2+(20\alpha^3-\psi\beta^2)y_0y_1+(20\beta^3-\psi\alpha^2)y_0y_2+10\alpha y_1^2+10\beta y_2^2+\psi y_1y_2$$

at $y_0 = x_4^2$, $y_1 = s^2$, $y_2 = t^2$. The Hessian matrix associated to the equation of the conic C_ψ thus determined is

$$\begin{pmatrix} 2(1+2\alpha^5+\psi\alpha^2\beta^2+2\beta^5) & 20\alpha^3-\psi\beta^2 & 20\beta^3-\psi\alpha^2 \\ 20\alpha^3-\psi\beta^2 & 20\alpha & \psi \\ 20\beta^3-\psi\alpha^2 & \psi & 20\beta \end{pmatrix}$$

with determinant

$$\Delta_\psi(\alpha,\beta) := 2\Big(400\alpha\beta(1-8\alpha^5-8\beta^5) + 1600\alpha^3\beta^3\psi - (1+32\alpha^5+32\beta^5)\psi^2 \Big).$$

We deduce that the pairs (α, β) satisfying $\Delta_\psi(\alpha,\beta) = 0$ in $\mathbb{A}^2_{\alpha,\beta}$ correspond to reducible conics C_ψ, and the components of such reducible conics correspond to conics contained in the quintic Q_ψ. Thus the space of conics contained in Q_ψ contains a subscheme isomorphic to a double cover of the vanishing set of Δ_ψ. In fact, the closure of such a subscheme is a component of the space of conics contained in Q_ψ.